Effect of Finite Ion Temperature and Modulational Instability of Ion-Acoustic Waves in Presence of an Inhomogeneous Plasma

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Using an extended version of K.B.M. method we have investigated the effect of finite ion temperature on ion-acoustic solitary waves. Modulational instability has been discussed in a frame work of nonlinear Schrödinger equation. Some numerical results are also given.

Introduction

Several authors have made attempts to examine the effect of density gradients on the behaviour of solitons.

Chen and Liu [1] have investigated the non linear propagation in an homogeneous medium in the model of non linear Schrödinger equation. Gell and Gomberoff [2] studied this in the frame work of Kdv model.

The effect of finite ion temperature on ion-acoustic solitary waves has been studied by Shivamoggi [3] in a frame work of Korteweg-de Vries model.

Here we have used the extended version of the K.B.M. method (see [4], [5]) to study the effect of finite ion temperature on ion-acoustic solitary waves and derive a non linear Schrödinger equation.

Unlike Shivamoggi we did not have to assume $p_0(x) = (h_0(x))$ from the beginning. We have also discussed the effect of the ion temperature and inhomogeneous terms on the critical values of the wave number for modulational instability.

It was pointed out by Kakutani and Sugimoto [4] that ion acoustic waves with short wave lengths are unstable. More precisely they are unstable for wave numbers $> k_c \simeq 1.471\,(\lambda_D)^{-1}$, where λ_D is the Debye length.

We find that this value is increased distinctly owing to the presence of finite ion temperature, but the increment is slow for the presence of an inhomogeneous term.

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Mathematical Method

The fundamental set of equations governing the propagation of non linear ion-acoustic waves for a collision-less plasma composed of a warm-ion fluid and hot isothermal electrons are (see [6] and [7]):

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,u) = 0\,, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\frac{\sigma}{n}\,\frac{\partial p}{\partial x} - \frac{\partial \psi}{\partial x}\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} + 3\,p\,\frac{\partial u}{\partial x} = 0\,, \tag{3}$$

$$\frac{\delta^2 \psi}{\delta x^2} = n_e - n\,, \tag{4}$$

$$\frac{1}{n_e}\,\frac{\delta n_e}{\delta x} = \frac{\partial \psi}{\partial x} = -E\,, \tag{5}$$

where n, n_e, u, E denote respectively the ion-density, the electron-density, the ion-fluid velocity and the electric field. σ is the ratio of ion temperature to electron temperature and p is the ion pressure.

All the variables have been normalized in the following way:

$$n = \frac{n_i'}{N_0'}\,,\quad n_e = \frac{n_e'}{N_0'}\,,\quad u = \frac{u'}{(K T_e'/m_i')^{1/2}}\,,$$

$$\psi = \frac{\Phi'}{K T_e'/e}\,,\quad x = \frac{x'}{\left(\dfrac{K T_e'}{4\pi n_0' e^2}\right)^{1/2}}\,,$$

$$t = t'\left(\frac{4\pi N_0' e^2}{m_i'}\right)^{1/2},\quad \sigma = \frac{T_i'}{T_e}\,,$$

$$p = \frac{p'}{N_0' K T_i'}\,,$$

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where n' is the number density, u' the flow density p' the ion pressure, T' the temperature, Φ' the electrostatic potential, and x' and t' are the space and time coordinates. The primed quantities mean unnormalized variables and the unprimed ones denote the normalized variables. i and e denote respectively the ion and electron and K denotes Boltzmanns constant.

Using (2), (4), (5) we have,

$$\frac{\partial n}{\partial x} - \left[\frac{\delta^3 u}{\delta \varkappa\, \delta t} + \frac{\delta^2}{\delta x^2}\left(u \frac{\delta u}{\delta x}\right) + \frac{\delta^2}{\delta x^2}\left(\frac{\sigma}{n}\frac{\delta p}{\delta x}\right)\right]$$
$$+ \left[n \cdot \frac{\delta u}{\delta t} + n u \cdot \frac{\delta u}{\delta \varkappa} + \sigma \frac{\delta p}{\delta x}\right]$$
$$- \left[\frac{\partial}{\partial x}\left(\frac{\sigma}{n}\frac{\partial p}{\partial x}\right) + \frac{\delta^2 u}{\delta x\, \delta t} + \frac{1}{2}\frac{\delta^2 (u^2)}{\delta x^2}\right]$$
$$\cdot \left[\frac{\sigma}{n}\frac{\partial p}{\partial x} + \frac{\delta u}{\delta t} + \frac{1}{2}\frac{\delta}{\delta x}(u^2)\right] = 0 \,. \tag{6}$$

Let us seek the solution of the above equation in the form (ε being a small parameter)

$$\begin{aligned} n &= n_0(x)\,[1 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \ldots]\,, \\ n &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots\,, \\ p &= p_0(x)\,[1 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \ldots]\,, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\frac{\partial n_0}{\partial t} = 0, \quad \frac{\partial p_0}{\partial t} = 0, \quad \frac{\partial p_0}{\partial x} = \varepsilon^2 \alpha_2, \\ &\frac{\partial n_0}{\partial x} = \varepsilon^2 \alpha_1 \,. \end{aligned} \tag{8}$$

If a, $\bar{a}$ denote the complex amplitudes of the plane wave solution of monochromatic emission at zero order, then we have

$$\begin{aligned} u_1 &= a\, e^{i\psi} + \bar{a}\, \bar{e}^{i\psi}\,, \\ n_1 &= \frac{k}{w}(a\, e^{i\psi} + \bar{a}\, \bar{e}^{i\psi})\,, \\ p_1 &= \frac{3k}{w}(a\, e^{i\psi} + \bar{a}\, \bar{e}^{i\psi})\,. \end{aligned} \tag{9}$$

The amplitudes a, $\bar{a}$ vary slowly as

$$\frac{\partial a}{\partial t} = \varepsilon A_1(a, \bar{a}) + \varepsilon^2 A_2(a, \bar{a}) + \ldots\,, \tag{10}$$

$$\frac{\partial a}{\partial x} = \varepsilon B_1(a, \bar{a}) + \varepsilon^2 B_2(a, \bar{a}) + \ldots \tag{11}$$

together with the complex conjugate of the above relations.

The dispersion relation that follows from the solutions (7) to (9) is as follows:

$$(k^2 - w^2) + \frac{3\sigma p_0 k^4}{n_0^2} + \frac{3\sigma p_0 k^2}{n_0} = \frac{k^2 w^2}{n_0}\,. \tag{12}$$

The second set of equations for $0(\varepsilon^2)$ gives the secular free condition

$$A_1 + V_g B_1 = 0\,, \tag{13}$$

where the group velocity V_g is given by

$$V_g = \frac{w^3}{k^3}\left(1 + \frac{3\sigma p_0 k^4}{n_0^2 w^2}\right)\left(1 - \frac{3\sigma p_0 k^2}{n_0 w^2}\right). \tag{14}$$

Then the order solutions are

$$u_2 = \xi\, e^{i\psi} + F_2 a^2 e^{2i\psi} + \mathrm{c \cdot c} + \gamma'\,, \tag{15}$$

$$n_2 = b_1 e^{i\psi} + \eta a^2 e^{2i\psi} + \mathrm{c \cdot c} + \gamma\,, \tag{16}$$

$$p_2 = 3\, b_1 e^{i\psi} + F_3 a^2 e^{2i\psi} + \mathrm{c \cdot c} + \gamma_p\,, \tag{17}$$

where

$$\xi = \frac{w}{k} b_1 + B_1\left[\frac{i}{k} - \frac{i V_g}{w}\right], \tag{18}$$

$$F_2 = \frac{w}{k}\eta - \frac{k}{w}\,, \tag{19}$$

$$F_3 = 3\eta + \frac{3k^2}{w^2}\,, \tag{20}$$

$$\eta = \frac{24k^4 + 2k^2 w^2 + 4k^2 n_0}{12\left(k^2 w^2 - \dfrac{3\sigma p_0 k^4}{n_0}\right)} + \frac{\sigma p_0 k^4\left[\dfrac{24k^2}{w^2 n_0} + \dfrac{18\sigma p_0^2 k}{n_0^2 w^2} - \dfrac{12}{n_0} + \dfrac{12}{w^2}\right]}{12\left(k^2 w^2 - \dfrac{3\sigma p_0 k^4}{n_0}\right)}\,, \tag{21}$$

and b_1, $\bar{b}_1$ (complex) and γ, γ', γ_p (real) are constants independent of w but dependent on a, $\bar{a}$. They are determined from the non secular conditions at higher orders. c.c. denotes complex conjugate.

Applying the non secularity conditions at $0(\varepsilon^3)$ we obtain

$$\gamma' = \frac{\left(\frac{2\,k\,n_0}{w} + w^2\,V_g\right) + \frac{6\,\sigma\,p_0\,k}{w}\left(1 + \frac{k}{w} - \frac{k\,w\,V_g}{n_0} + \frac{3}{2}\,\frac{k^2\,\sigma\,p_0\,V_g}{n_0^2\,w}\right)}{\left(V_g^2 - 1 - \frac{3\,\sigma\,p_0}{n_0}\right)n_0}\,a\,a + \lambda_1\,, \tag{22}$$

$$\gamma = \frac{\left(\frac{2\,k\,V_g\,n_0}{w} + w^2\right) + \frac{6\,\sigma\,p_0\,k}{w\,V_g}\left(\frac{k}{w} - \frac{k\,w\,V_g}{n_0} + \frac{3}{2}\,\frac{p_0\,\omega\,k^3\,V_g}{n_0^2\,w}\right)}{\left(V_g^2 - 1 - 3\,\frac{\sigma\,p_0}{n_0}\right)n_0}\,a\,\bar{a} + \lambda_2\,, \tag{23}$$

$$\gamma_p = \left[\left(\frac{6\,k\,n_0\,V_g}{w} + 3\,w^2 + \frac{6\,k^2}{w^2}\,(V_g^2 - 1) + \frac{18\,\sigma\,p_0\,k}{w\,V_g}\left(\frac{k}{w} - \frac{k\,V_g}{w} - \frac{k\,V_g\,w}{n_0}\right.\right.\right.$$
$$\left.\left.\left. + \frac{3}{2}\,k^3\,\frac{\sigma\,p_0\,V_g}{n_0^2\,w}\right)\right]a\,\bar{a}\Big/\left[\left(V_g^2 - 1 - \frac{3\,\sigma\,p_0}{n_0}\right)n_0\right] + \lambda_3\,, \tag{24}$$

where λ_1, λ_2, λ_3 are arbitrary constants. When $\sigma = 0$, γ', γ conform to the parameters c_1, c_2 (see (30)) of [4]).

Before we derive the nonlinear Schrödinger equation we find that elimination of the resonant terms gives

$$i\,(A_2 + V_g\,B_2) + P\left[B_1\,\frac{\partial B_1}{\partial a} + \bar{B}_1\,\frac{\partial B_1}{\partial a}\right] = Q\,a^2\,\bar{a} + R\,a\,, \tag{25}$$

where

$$P = \left[-\frac{3\,w^3}{2\,n_0\,k^4} - \frac{3}{2\,n_0\,k}\left(C_V - \frac{3\,\sigma\,p_0\,k}{n_0}\right)\right.$$
$$\left. + \frac{1}{2\,k^2}\,(1 + k^2/n_0)\,(1/k - V_g/w)\left(C_V + \frac{3\,\sigma\,p_0\,w}{n_0\,k}\right)\right]\Big/\left[\left(w^2 - \frac{3\,\sigma\,p_0\,k^2}{n_0}\right)\right], \tag{26}$$

$$Q = \left[\left\{\frac{3}{2}\,\frac{w}{n_0} + \frac{w}{k^2} - \frac{w^3}{k^2\,n_0} + \frac{\sigma\,p_0}{2\,n_0^2\,w}\left(3\,k^2 + 12\,w^2 - 6\,n_0 - \frac{18\,\sigma\,p_0\,k^2}{n_0}\right)\right\}\eta\right.$$
$$+ \left\{\left(\frac{1}{2\,w} - \frac{k^2}{n_0\,w} + \frac{3}{2}\,\frac{w}{n_0}\right) - \frac{\sigma\,p_0\,k^2}{2\,n_0^2\,w^2}\left(21\,k^2 + \frac{9\,\sigma\,p_0\,k^2}{n_0} + 6\,w^3 + 15\,n_0\right)\right\}$$
$$+ \gamma_s'\left\{\frac{1}{k}\,(1 + k^2/n_0) + \frac{3}{2}\,\frac{\sigma\,p_0\,k}{w^2\,n_0^2}\,(k^2 + n_0)\,(1 - w)\right\}$$
$$\left. + \gamma_s\left\{\frac{w}{2\,n_0} - \frac{3}{2}\,\frac{\sigma\,p_0}{n_0^2\,w}\,(n_0 + 2\,k^2)\right\} + \frac{3\,\sigma\,k_0}{2\,n_0^2\,w}\,\gamma_{ps}\,(k^2 - n_0)\right]\Big/\left[w^2 - \frac{3\,\sigma\,p_0\,k^2}{n_0}\right], \tag{27}$$

$$R = \left[-\alpha_1\left(\frac{1}{n_0\,k\,w} + \frac{3\,\sigma\,p_0\,k}{n_0^3\,w}\right)i - \alpha_2\,i\left(\frac{9\,\sigma\,k}{2\,n_0\,w} + \frac{2\,\sigma}{n_0\,k\,w}\right.\right.$$
$$\left. + \frac{3}{2}\,\frac{\sigma^2\,p_0\,k}{n_0^3\,w} - \frac{\sigma\,w}{2\,n_0^2\,k}\right) + \left\{\frac{k}{w^2} + \frac{3\,\sigma\,p_0\,k}{2\,n_0\,w}\left(\frac{3}{w} - 1\right)\left(\frac{k^2}{n_0} + 1\right)\right\}\lambda_1$$
$$\left. + \left\{\frac{w}{2\,n_0} - \frac{3\,\sigma\,p_0}{2\,n_0^2\,w}\,(n_0 + 2\,k^2)\right\}\lambda_2 + \frac{3}{2}\,\sigma\,p_0\left(\frac{k^2}{n_0^2\,w} + \frac{1}{n_0\,w}\right)\lambda_3\right]\Big/\left[\left(w^2 - \frac{3\,\sigma\,p_0\,k^2}{n_0}\right)\right], \tag{28}$$

here γ_s', γ_s, γ_{ps} are the coefficients of $a\,\bar{a}$ in γ', γ and γ_p respectively, and

$$C_V = \left(\frac{-3\,\sigma\,p_0\,w}{n_0\,k} + \frac{3\,\sigma\,p_0\,k\,w}{n_0^2} - \frac{9\,\sigma^2\,p_0^2\,k^3}{n_0^3\,w} \right).$$

Introducing the coordinate transformation defined as

$$\xi = \frac{1}{\varepsilon}(x_2 - V_g\,t_2) = \varepsilon(x - V_g\,t)\,, \tag{29}$$

$$\tau = t_2 = \varepsilon^2\,t\,,$$

(25) can be reduced to the nonlinear Schrödinger equation

$$i\,\frac{\partial a}{\partial \tau} + P\,\frac{\delta^2 a}{\delta \xi^2} = Q\,|a^2|\,a + R\,a\,. \tag{30}$$

When $\sigma = \alpha_2 = 0$, $\alpha_1 = 0$, P, Q, R conform to the form given by Kakutani and Sugimoto [4].

Discussion and Conclusion

It was shown by Tanuiti and Yajima [8] and also Hasimoto and Ono [9] that the plane wave solution of the nonlinear Schrödinger equation is modulationally unstable if $P\,Q < 0$.

Now, if k_c denotes the value such that $P\,Q \gtrless 0$ according as $k \gtrless k_c$ then k_c is called the critical wave number.

To see the effect of finite ion temperature and inhomogeneity we take $n_0(x)$, $p_0(x)$ of the form $p_0(x) = n_0(x) = n_0(1 + a\,x)$, where a is a small number given by $a = 1/L$ and numerically evaluate k_c for some selected values of a:

Values of k_c.

a \ σ	0	0.001	0.01
0	1.471	1.605	1.650
0.02	1.485	1.603	1.672
0.025	1.490	1.627	1.678
0.033	1.50	1.635	1.686

Our results for $\sigma = 0$, $a \neq 0$ differ from those of Durrani et al. [5]. But it must be mentioned that their results do not conform to those obtained by Kakutani and Sugimoto in the limit $\lambda \to 1$. Also it seems that their values for k_c do not conform to their own formula for Q as given in [5].

The above table shows that the values of k_c increase as σ increases. The same holds for the inhomogeneity parameter a, but in the latter case the rate of increase is slow and proportional to $\sqrt{n_0}$.

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